q calculus and entropy in nonextensive statistical physics

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(Received 6 March 1998)

The connection between Tsallis entropy for a multifractal distribution and Jackson's q derivative is established. Based on this derivation and definition of a homogeneous function, a q analog of Shannon's entropy is defined. Tsallis entropy can also be accommodated in this formalism. Pseudoadditivity of the q entropies is proved. We also define a q analog of Kullback relative entropy. The implications of the lattice structure beneath the q calculus are highlighted in the context of the q entropy. [S1063-651X(98)08809-6]

PACS number(s): 05.50.+q, 05.70.−a, 02.20.−a

I. INTRODUCTION

There are many nonextensive systems in nature, like gravitational systems [1], magnetic systems [2], Levy-like anomalous diffusion phenomena [3], etc., which are untractable within the conventional Boltzmann-Gibbs (BG) statistics. In such cases, it is impossible to obtain well behaved expressions for response functions of thermodynamic quantities, which can provide comparisons with experimental results. The main reason for this failure is that BG statistics is an extensive (or additive) formalism. Nonextensive Tsallis thermostatistics (NTT) [4] was proposed as a formalism suitable for treating nonextensivity of physical systems [5]. It has been applied with success to many different problems. A few examples are finite mass for stellar polytropes [6], Levy-like anomalous diffusion [7], calculation of the specific heat of nonionized hydrogen atom [8], velocity distribution of galaxy clusters [9], and linear response theory [10]. For detailed reviews on NTT, formalism as well as applications, see [11]. A complete list of works on the formalism is also available on the internet [12]. An important fact about NTT is that an entire formalism of thermodynamics can be extended within NTT. One nonextensive quantity that plays a useful role in this context is Tsallis entropy [13]. Given a probability distribution $\{p_i\}_{i=1,\dots,W}$ where i is the index for system configuration, Tsallis entropy is given by

$$S_q^T = \frac{1-\sum_{i=1}^{W}(p_i)^q}{q-1}, \tag{1}$$

where q is a real parameter, assumed to be positive. W is the number of accessible configurations and Boltzmann's constant k_B has been set equal to unity. Tsallis entropy has some important properties such as positivity, concavity, and pseudoadditivity. As $q\to 1$, $S_q^T \to -\Sigma p_i \ln p_i$, which is Shannon entropy. Thus parameter q describes deviations of Tsallis entropy from Shannon entropy. On a different side, quantum algebras [14], q analysis and q special functions [15] and q-deformed physical theories [16] have been the subject of great attention in the last decade. Here the serious interest in q deformation lies in more than just modifying the undeformed theory at the phenomenological level. Some theories have been shown to contain q-deformed structure inherently in themselves. Thus in [17,18], underlying a q-deformed formalism, there is invariably a lattice structure. The deformation parameter can be given in terms of the size of lattice elementary cell. The Barnett-Pegg formalism for the rotation angle operator is another example [19] where the q parameter is related to the dimension of the representation. It was pointed out recently that q analysis is naturally suited for study of fractal and multifractal sets [20]. An important feature of these theories is the presence of one (or more) deformation parameter q, which describes deviation from standard undeformed theories. Usually, for $q\to 1$, the formalism reverts to the standard one. Recently, Tsallis noted [13] a similarity between q numbers used in q-deformed theories and the entropy of Eq. (1). Notably the pseudoadditive properties of both quantities are alike. Abe [21] suggested the use of Jackson's q derivative to form a link between the q calculus and Tsallis entropy. Erzan [22] has shown that the nonhomogeneity relation obeyed by the nonextensive free energy can be expressed in terms of the q-difference operator. Also in [23] the nonextensivity of the classical set theory was shown to have a q-oscillator structure. These works suggest that the property of nonextensivity has deeper roots in q-deformation structure, though a complete understanding of this relation is lacking [24]. This paper attempts to further narrow this gap. We will seek to accommodate entropy, especially Tsallis entropy, in a q-analytic framework. An interpretation highlighting the lattice structure beneath a q framework [17] of entropy will also be given.

The plan of the paper is as follows: in Sec. II, we establish the relation between Jackson's q derivative and entropy for a multifractal probability distribution. We will use a slightly different definition for entropy than used in [21]. In Sec. III, we propose a more general definition of q entropy based on a definition of a homogeneous function. It is shown that pseudoadditivity of this entropy follows from pseudoadditivity of q numbers. Moreover, Tsallis entropy can also be accommodated in this definition. In Sec. IV, we define the q analog of Kullback relative entropy based on the above considerations. Lastly, before concluding, the lattice structure behind the q-calculus framework of entropy is highlighted.

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A. Shannon entropy for multifractals

Originally, Tsallis' proposal was aimed at accommodating scale invariance in systems with multifractal properties to the thermodynamic formalism. Let us also start with a multifractal probability distribution. Fractal attractors of nonlinear mappings are of this type. According to the box counting algorithm, we divide the d-dimensional phase space into d-dimensional cubes (hereby called boxes) with side length R. Let the number of microstates W represent the number of boxes with nonzero probability. For finite value of R, we assume a local scaling for the probabilites,

$$p_i = R^{\alpha_i}, \tag{2}$$

where $\{\alpha_i\}_{i=1,\ldots,W}$ is the set of crowding indices. Thus, in a manner similar to suggested in [21], we can write Shannon's entropy for the given probability distribution as

$$-\sum_i \alpha_i \frac{d}{d\alpha_i} p_i = -\sum_i p_i \ln p_i . \tag{3}$$

Note, however, that the variable α_i here can be given a suitable interpretation, which was a dummy variable in [21].

B. q calculus and Tsallis entropy

If we replace the ordinary derivative in Eq. (3) with Jackson's q derivative [25], we get

$$-\sum_i \alpha_i \mathcal{D}^q_{\alpha_i} p_i = \frac{1-\sum_{i=1}^{W} (p_i)^q}{q-1}, \tag{4}$$

which is Tsallis entropy. Instead of Jackson's derivative, if we use the symmetric q derivative that has $q \leftrightarrow q^{-1}$ invariance, we obtain the alternate entropy suggested in [21]. In the following, we concentrate on the entropy based on Jackson's derivative. It will be clear in Sec. III that the form (4) helps one to visualize a more general definition for entropy.

We argue that although use of ordinary derivative with respect to α_i in Eq. (3) is mathematically correct, it is not proper in an *operational* sense. Note that in the limit, $R \to 0$, $\alpha_i \to \alpha(x)$ (x represents the position coordinate in phase space). Then we have an entire spectrum of different crowding indices $\alpha(x)$. But in practice, as in computation, a fractal object is defined within a finite range of length scales, i.e., $R_{\min} < R < R_{\max}$, where $R_{\min}$ is large compared with average interatomic spacing and $R_{\max}$ is small compared with the geometric extent of the object. Thus the number as well as size of boxes is finite. In this situation, the α_i's may not be continuously distributed. So it makes more sense to use a discrete derivative to define entropy than a continuous one in Eq. (3). The suitability of the q derivative for multifractals has been stressed in [20].

II. GENERALIZED q ENTROPY

In this section, we generalize the definition of entropy. Consider an arbitrary probability distribution $\{p_i\}$ where $p_i(\alpha_i)$ is homogeneous function of degree a_i and α_i is not necessarily a scaling index. Then by definition

$$\alpha_i \mathcal{D}^q_{\alpha_i} p_i = [a_i] p_i , \tag{5}$$

where $[a_i] = (q^{a_i}-1)/q-1$ is the Jackson q number. Then we define the q entropy as

$$-\sum_i \alpha_i \mathcal{D}^q_{\alpha_i} p_i = -\sum_i [a_i] p_i . \tag{6}$$

As $q \to 1$, we get

$$-\sum_i \alpha_i \frac{d}{d\alpha_i} p_i = -\sum_i a_i p_i . \tag{7}$$

If we identify $a_i = \ln p_i$ in Eq. (7), we get Shannon entropy on the right-hand side. Alternatively, if we set α_i as the local scaling index, we again obtain Shannon entropy as defined in Eq. (3). Similarly, by setting $a_i = \ln p_i$ in Eq. (6) so as to obtain Shannon entropy in the limit $q \to 1$, we get the q deformed analog of Shannon entropy

$$S'_q = -\sum_i [\ln p_i] p_i . \tag{8}$$

Defining q entropy in the language of homogeneous functions provides us with the q analog $[\ln p_i]$, of bit of information $a_i = \ln p_i$ [26]. In other words, the definition (5) gives $\Sigma_i [\ln p_i] p_i$ as a q analog of Shannon information. Note that it is nontrivial to say that the q analog of bit number $-\ln p_i$ is $-[\ln p_i]$, but it actually follows from definition (5) of a homogeneous function as well as power law behavior of probability distribution (2).

Now we show that Tsallis entropy can also be accommodated in the definition of q entropy (6). Taking α_i as a scaling index, then equality of Eqs. (4) and (6) gives

$$[a_i] = \frac{q^{a_i}-1}{q-1} = \frac{(p_i)^{q-1}-1}{q-1}, \tag{9}$$

which gives

$$a_i = \frac{q-1}{\ln q} \ln p_i . \tag{10}$$

Thus we can alternatively define Tsallis entropy as the *negative of the mean of* $[a_i]$*'s over the probability distribution*, where a_i is given by Eq. (10).

In Fig. 1, we compare the maximum values of the respective entropies for equiprobability distribution of states ($W = 50$). Only entropy values for $q < 1$ and not for $q > 1$ appear to be physically meaningful, as discussed below in Sec. V. As is clear from Fig. 2, S'_q is also a concave function.

Note that if we write $q = 1 + \delta$, then in Eq. (10), for $\delta \ll 1$, $\ln q = \delta$, which implies that Tsallis and q entropy are equal there. The two entropies begin to differ when higher order terms in the expansion of $\ln q$ become significant. In Fig. 3, we show this difference when a second order term is included.

Next we show how the pseudoadditive property of S'_q as well as of Tsallis entropy follows very simply from a similar property of q numbers. For this purpose, consider two independent subsystems I and II, described by normalized prob-

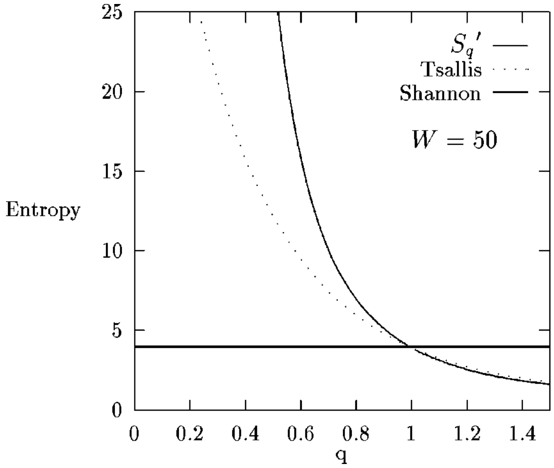

FIG. 1. Comparison of entropies for equiprobability distribution at $W=50$, as a function of parameter q.

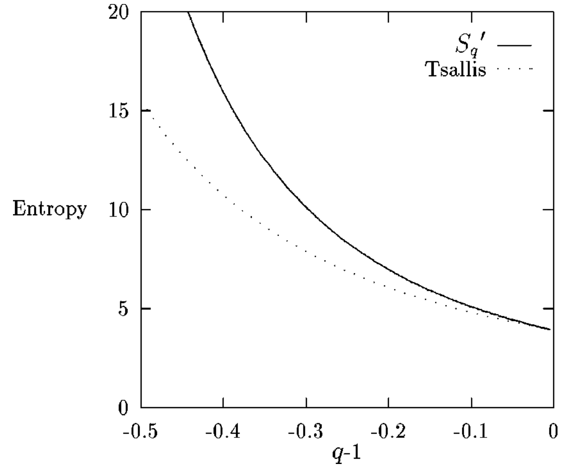

FIG. 3. Deviation of maximum Tsallis entropy from maximum S_q' when the second order term in $\delta=q-1$ is included in $\ln q$, for $W=50$.

ability distributions $\{p_i\}$ and $\{p_j\}$, respectively. Then the q entropy of the combined system may be written as

$$S_q'(\mathrm{I}+\mathrm{II}) = -\sum_{i,j} [\ln p_{ij}] p_{ij},$$

$$= -\sum_{i,j} [\ln p_i + \ln p_j] p_i p_j$$

$$= S_q'(\mathrm{I}) + S_q'(\mathrm{II}) + (1-q) S_q'(\mathrm{I}) S_q'(\mathrm{II}), \quad (11)$$

where we have made use of the pseudoadditive property of Jackson's q numbers and the fact that probability distributions are normalized. It is interesting to note that pseudoadditivity of Tsallis entropy emerges because this entropy can be looked upon as q entropy where now a_i is given by relation (10).

III. q ANALOG OF KULLBACK ENTROPY

Kullback relative entropy [27] measures the information gained in going from a probability distribution p^0 to another one p. In the literature, generalized Kullback entropy is attracting attention within the framework of NTT [28]. We can also define the q analog of Kullback relative entropy based on the definition of q entropy (6).

Consider the difference $[a_i] - [a_i^0]$, where $a_i = \ln p_i$ and $a_i^0 = \ln p_i^0$. The average weighted against the new probability distribution gives the q analog of the Kullback relative entropy,

$$K_q(p,p^0) = \sum_i p_i([a_i] - [a_i^0]). \quad (12)$$

In the limit of $q \to 1$, we get the standard Kullback relative entropy. Using $K_q(p^0,p) = \Sigma_i p_i^0([a_i^0] - [a_i])$, we obtain the q analog of the symmetric sum,

$$D_q(p,p^0) = K_q(p,p^0) + K_q(p^0,p)$$

$$= \sum_i ([a_i] - [a_i^0])(p_i - p_i^0). \quad (13)$$

Each term in the sum is positive and is zero if $p_i = p_i^0$. Thus this function appears suitable for a metric in the functional space of probability distributions [29].

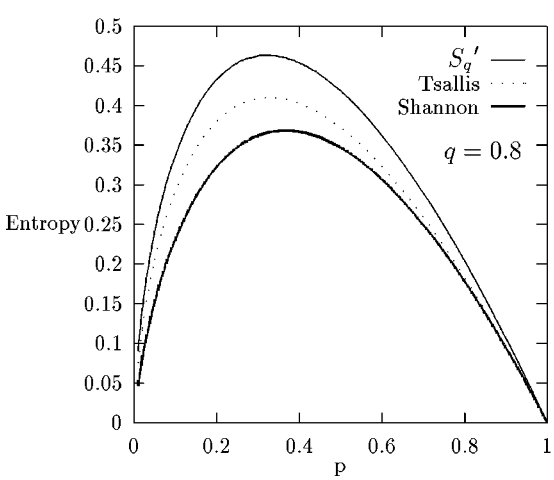

FIG. 2. Concavity of different entropies. q parameter is set at 0.8.

IV. q ENTROPY AND LATTICE STRUCTURE

Finally, we remark on the lattice structure that underlies the q calculus framework of entropy. A natural lattice already exists, because we partition the phase space into boxes of equal size R. The lattice constant R can be identified with $|q-1|$. Thus $q \to 1$ limit also implies $R \to 0$. The finite size of the boxes causes coarse graining of the phase space, as a result the information we would have about the structure of the distribution is also coarse grained. Thus, we emphasize that *the values of q entropies should be greater than the Shannon entropy*, which is shown here as the limit of $q \to 1$ case. We note that both Tsallis and q entropy (9) satisfy this condition for $q<1$ (Fig. 1). The divergence of q entropy can also be explained because as $q \to 0$, the size of the boxes increases, which causes more loss of information and thus gain in entropy.

As a physical example, in the sandpile model of selforganized criticality [30], the size of sand grain can represent the lattice constant $|q-1|$. Changing the lattice constant can change the profile of the sandpile, which is equivalent to using wet/dry sand or larger/smaller grains. Nonextensivity can be important in this system because of the presence of self-similarity for a range of spatial and temporal scales [31].

One can look at the lattice structure of Tsallis entropy from a different viewpoint. By definition, the homogeneous function p_i has a discrete spectrum that can be written as $p_0, qp_0, q^2p_0, \ldots, q^{W-1}p_0$, where W is the number of microstates and we assume $q<1$. In p_i space, we can speak in terms of the so-called q lattice. In fact, relation (10) is the exact transformation considered in [17], which maps from the q lattice (in p_i space) to an equidistant lattice (in a_i space), with lattice constant of $\ln q$. Moreover, due to normalization of probability distribution, we have a constraint that helps to determine the initial point p_0. Thus

$$p_0+qp_0+q^2p_0+\cdots+q^{W-1}p_0=1, \tag{14}$$

which gives $p_0=1/[W]$. We have used the expansion property of the q number, $[W]=1+q+q^2+\cdots+q^{W-1}$. Thus as $q\rightarrow 1$, the distribution tends to the equiprobability distribution.

V. CONCLUSION AND OUTLOOK

We have presented a q generalization of the concept of entropy. Tsallis entropy, which specifically deals with nonextensive systems, can also be accommodated in this formalism. We have also defined the q analog of Kullback relative entropy, which is another quantity of interest, especially for a q-deformed metric in the functional space of probability distributions and for statistical inference. Finally, it is argued that the lattice structure that arises from the box-counting algorithm provides the required lattice, which is inherently present behind a q-calculus framework.

The applicability of q entropy S_q', in physical problems appears to be of immediate importance. Of course, for $q=1+\delta$ and up to $O(\delta^2)$, the q entropy and Tsallis entropy are the same. But for farther deviations of q from unity, they give different results.

The definition of entropy in terms of homogeneous functions can be useful for thermodynamic systems with discrete dilatation symmetries, such as spin systems on hierarchical lattices [32] (see also [22]). Again in thermodynamic context, nonadditivity of bit cumulants of different subsystems indicates correlations between those subsystems [26]. Broadly speaking, suppression of correlations without changing $\{p_i\}$ causes increase of entropy. We have argued (Sec. V) that coarse graining of the probability distribution should imply rise in entropy. This poses an interesting question: How does a q description of entropy affect correlations between subsystems? The q analogs of bit numbers discussed in Sec. III may play a useful role here.

ACKNOWLEDGMENTS

The author would like to gratefully acknowledge Sumiyoshi Abe for his encouraging remarks and useful discussions, as well as making Refs. [24] and [28] available.

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